B.Sc (Physics) Part-I, Paper-I, Group-B

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Generalized momentum and Cyclic Coordinates;

To define the generalized momentum, we consider that a particle is moving with velocity $\dot{x}$ along x-direction.

Now the Kinetic energy of the particle is given by

$$T = \frac{1}{2} m \dot{x}^2 \quad \text{--- (1)}$$

Next, the derivative w.r.to $\dot{x}$ is given as

$$\frac{\partial T}{\partial \dot{x}} = m\dot{x} \quad \text{--- (2)}$$

From (2) we see that $\frac{\partial T}{\partial \dot{x}} = p = m\dot{x}$

Now if we consider that potential is not a function of velocity $\dot{x}$, then $\frac{\partial V}{\partial \dot{x}} = 0$

Therefore

$$\frac{\partial}{\partial \dot{x}}(L) = \frac{\partial}{\partial \dot{x}}(T-V) = \frac{\partial T}{\partial \dot{x}} - \frac{\partial V}{\partial \dot{x}} = p - 0$$

or

$$\boxed{p = \frac{\partial L}{\partial \dot{x}}} \quad \text{--- (3)}$$

we can generalize the above equation for a set of generalized coordinates q_k $(k = 1, 2, \dots n)$

$$\boxed{p_k = \frac{\partial L}{\partial \dot{q}_k}} \quad \text{——— (4)}$$

$p_k \rightarrow$ conjugate momentum to coordinate q_k

~~or~~ we also call p_k as canonical momentum.

Now the Lagrange's equations for conservative system is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$$

from (4)

$$\frac{d(p_k)}{dt} - \frac{\partial L}{\partial q_k} = 0$$

or $$\dot{p}_k = \frac{\partial L}{\partial q_k} \quad \text{——— (5)}$$

Next, if we assume that ~~L does~~ the coordinate q_k does not appear explicitly in the expression of L, then we can write from eq (5) as

$$\dot{p}_k = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_k}\right] = \frac{\partial L}{\partial q_k} = 0$$

or $$\dot{p}_k = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0$$

Integrating (5)

$$\boxed{p_k = \frac{\partial L}{\partial \dot{q}_k} = \text{Constant}} \quad \text{——— (6)}$$

From equation ⑥, we see that when a particular coordinate q_k does not appear explicitly in the Lagrangian function, the generalized momentum p_k is a constant of motion. The coordinate q_k is called cyclic coordinate or ignorable coordinate.

Example of generalized momentum / cyclic / ignorable coordinate –

We consider here motion of a particle in a central force field. The Lagrangian function in plane polar coordinate is given by

$$L = T - V = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - V(r) \quad \text{—⑦}$$

In eq. ⑦, we see that the coordinate 'θ' does not appear in L.

{ In above expression velocity in polar coordinate is given by $v^2 = \dot{r}^2 + r^2\dot{\theta}^2$ }

Thus, θ is cyclic coordinate or ignorable coordinate.

The corresponding generalized momentum p_θ is given by

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} = \text{constant} \quad \text{—⑧}$$

Angular momentum and is constant of motion in time

From ⑧ we conclude that angular momentum of the system is conserved for a central force problem.